

RESTRICTION OF CHARACTERS AND PRODUCTS OF CHARACTERS

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ABSTRACT. Let G be a finite p -group, for some prime p , and $\psi, \theta \in \text{Irr}(G)$ be irreducible complex characters of G . It has been proved that if, in addition, ψ, θ are faithful characters, then the product $\psi\theta$ is a multiple of an irreducible or it is the nontrivial linear combination of at least $\frac{p+1}{2}$ distinct irreducible characters of G . We show that if we do not require the characters to be faithful, then given any integer $k > 0$, we can always find a p -group G and irreducible characters Ψ and Θ such that $\Psi\Theta$ is the nontrivial combination of exactly k distinct irreducible characters. We do this by translating examples of decompositions of restrictions of characters into decompositions of products of characters.

1. INTRODUCTION

Let G be a finite group. Denote by $\text{Irr}(G)$ the set of irreducible complex characters of G . Let ψ and θ be characters in $\text{Irr}(G)$. Since a product of characters is a character, the product $\psi\theta$, where $\psi\theta(g) = \psi(g)\theta(g)$ for all $g \in G$, is a character. Then the decomposition of the character $\psi\theta$ into its distinct irreducible constituents $\phi_1, \phi_2, \dots, \phi_k \in \text{Irr}(G)$ has the form

$$\psi\theta = \sum_{i=1}^k n_i \phi_i$$

where $k > 0$ and $n_i = (\psi\theta, \phi_i) > 0$ is the multiplicity of ϕ_i in $\psi\theta$ for each $i = 1, \dots, k$. Let $\eta(\psi\theta) = k$ be the number of distinct irreducible constituents of the character $\psi\theta$.

Given any subgroup H of G , we denote by χ_H the restriction of the character χ of G to H .

Let G be a finite p -group, where $p > 2$ is a prime number. Given any two faithful characters $\psi, \theta \in \text{Irr}(G)$, in Theorem A of [1] is proved that the product $\psi\theta$ is either a multiple of an irreducible, i.e. $\eta(\psi\theta) = 1$, or $\psi\theta$ is the linear combination of at least $\frac{p+1}{2}$ distinct irreducible constituents, i.e. $\eta(\psi\theta) \geq \frac{p+1}{2}$. Thus there is not 5-group G with faithful characters $\psi, \theta \in \text{Irr}(G)$ such that $\psi\theta$ has exactly two distinct irreducible constituents. But can we find a 5-group P with characters $\psi, \theta \in \text{Irr}(P)$ such that $\psi\theta$ has exactly two distinct irreducible constituents? The answer is yes. Moreover

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Theorem A. Fix a prime number p . Let $k > 0$ and $n_i > 0$, $i = 1, \dots, k-1$, be integers. Choose $r > 0$ such that $p^r > \sum_{i=1}^{k-1} n_i$. Let $n_k = p^r - \sum_{i=1}^{k-1} n_i$. Then there exists a finite p -group G with characters $\Psi, \Theta \in \text{Irr}(G)$ such that

$$\Psi\Theta = \sum_{i=1}^k n_i \Phi_i,$$

where $\Phi_1, \Phi_2, \dots, \Phi_k$ are distinct irreducible characters in $\text{Irr}(G)$ and $n_i = (\Psi\Theta, \Phi_i)$. Thus $\eta(\Psi\Theta) = k$.

The main key for the previous result is the following

Theorem B. Let P be a finite p -group, $Q < P$ be a subgroup of P and $\psi \in \text{Irr}(Q)$. Assume that

$$(i) \psi_Q = \sum_{i=1}^k n_i \phi_i,$$

where $\phi_1, \phi_2, \dots, \phi_k$ are distinct irreducible characters in $\text{Irr}(Q)$ and $n_i = (\psi_Q, \phi_i) > 0$ for each $i = 1, 2, \dots, k$. Then there exists a p -group G with characters $\Psi, \Theta \in \text{Irr}(G)$ such that

$$(ii) \Psi\Theta = \sum_{i=1}^k n_i \Phi_i,$$

where $\Phi_1, \Phi_2, \dots, \Phi_k$ are distinct irreducible characters in $\text{Irr}(G)$ and $n_i = (\Psi\Theta, \Phi_i) > 0$ for each $i = 1, 2, \dots, k$.

The previous result allows us to translate all examples of decompositions of restrictions of characters into decompositions of products of characters. Since the product $\psi\theta$ of two characters ψ, θ of a group G can be regarded as the restriction of the character $\psi \times \theta$ of the direct product group $G \times G$ to the diagonal subgroup $D(G) = \{(g, g) | g \in G\}$, which is another copy of G inside $G \times G$, the converse also holds.

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2. PROOFS

Lemma 2.1. Given any finite p -group P and any subgroup Q of P , there exists an elementary abelian p -group E and $\lambda \in \text{Lin}(E)$ such that P acts on E as automorphisms and Q is the stabilizer of λ in P .

Proof. Let S be a finite set on which P acts transitively with Q as the stabilizer of one point $s \in S$. Then extend the action of P on S to one of P as automorphisms of the elementary abelian p -group $E = \langle S \rangle$ with S as a basis. Finally, take λ to be any linear character on E sending s to some primitive p -th root of unity, and every $s' \in S - \{s\}$ to 1. \square

Notation. We will denote by 1_G the trivial character of G . We will also denote by $\text{Lin}(G)$ the set of linear characters of G . Also we denote by $1 \in G$ the identity element of G .

Proof of Theorem B. Let E be an elementary abelian p -group and $\lambda \in \text{Lin}(E)$ such that P acts on E as automorphisms of E and the stabilizer P_λ of λ in P is Q . We define G to be the semi-direct product $P \ltimes E$. We identify both P and E with their images in G . So

$$G = PE = P \ltimes E.$$

We define Ψ to be the inflation of $\psi \in \text{Irr}(P)$ to an irreducible character of G with the values

$$(2.2) \quad \Psi(\sigma\tau) = \psi(\sigma)$$

for any $\sigma \in P$ and $\tau \in E$.

Because Q fixes λ , the factor group $QE/\text{Ker}(\lambda)$ is naturally isomorphic to $Q \times (E/\text{Ker}(\lambda))$. It follows that there is a unique character $\phi_i\lambda \in \text{Irr}(QE)$, for each $i = 1, 2, \dots, k$, such that

$$(\phi_i\lambda)(\sigma\tau) = \phi_i(\sigma)\lambda(\tau)$$

for any $\sigma \in Q$ and $\tau \in E$. The characters $\phi_i\lambda$, for $i = 1, 2, \dots, k$, are clearly distinct and lie over λ . Because $QE = P_\lambda E$ is the stabilizer G_λ of λ in G , Clifford theory tells us that the characters

$$\Phi_i = (\phi_i\lambda)^G$$

induced by these $\phi_i\lambda$ are distinct and lie in $\text{Irr}(G)$.

The trivial character 1_Q on Q also defines an irreducible character $1_Q\lambda$ of QE , with the values

$$(1_Q\lambda)(\sigma\tau) = \lambda(\tau)$$

for any $\sigma \in Q$ and $\tau \in E$. This also induces an irreducible character

$$(2.3) \quad \Theta = (1_Q\lambda)^G \in \text{Irr}(G).$$

We now prove that

$$\Psi\Theta = \sum_{i=1}^k n_i \Phi_i.$$

To see this, we start with the formula

$$\Psi\Theta = \Psi(1_Q\lambda)^G = (\Psi_{QE}(1_Q\lambda))^G,$$

which holds because Θ is induced in (2.3) from the character $1_Q\lambda$ of QE . Since Ψ is inflated (2.2) from $\psi \in \text{Irr}(P)$, its restriction Ψ_{QE} is inflated from the character ψ_Q of $Q \simeq QE/E$. It follows that $\Psi_{QE}(1_Q\lambda)$ is precisely the product character $\psi_Q\lambda$. In view of (i) this product character is

$$\Psi_{QE}(1_Q\lambda) = \psi_Q\lambda = \sum_{i=1}^k n_i \phi_i\lambda.$$

Hence

$$\Psi\Theta = \left(\sum_{i=1}^k n_i \phi_i\lambda\right)^G = \sum_{i=1}^k n_i (\phi_i\lambda)^G = \sum_{i=1}^k n_i \Phi_i.$$

Thus (ii) holds. \square

Example 2.4. Fix a prime number p . Let $k > 0$ and $n_i > 0$, $i = 1, \dots, k-1$, be integers. Choose integers $r, t > 0$ such that $p^r > \sum_{i=1}^{k-1} n_i$ and $p^t \geq k$. Let Z_{p^t} be a cyclic group of order p^t . Let $N = Z_{p^t} \times Z_{p^t} \times \cdots \times Z_{p^t}$ be the direct product of p^r copies of Z_{p^t} . Thus $|N| = (p^t)^{p^r}$. Observe that Z_{p^r} acts on N by permuting the entries of N . Set $P = Z_{p^r} \ltimes N = Z_{p^r} N$ and thus $P = Z_{p^t} \wr Z_{p^r}$ is the wreath product of Z_{p^t} by Z_{p^r} .

Fix a generator c in Z_{p^t} and $\alpha \in \text{Lin}(Z_{p^t})$ such that $\alpha(c)$ is a primitive p^t -th root of unity. Set $\lambda = (\alpha, 1_{Z_{p^t}}, \dots, 1_{Z_{p^t}}) \in \text{Lin}(N)$. We can check that the stabilizer P_λ of λ in P is N , and so $\psi = \lambda^P \in \text{Irr}(P)$ is a character of degree p^r . Observe also that $\psi_N = \sum_{i=1}^{p^r} \lambda_i$, where $\lambda_i = (1_{Z_{p^t}}, \dots, 1_{Z_{p^t}}, \alpha, 1_{Z_{p^t}}, \dots, 1_{Z_{p^t}}) \in \text{Lin}(N)$ is the character with α in the i -th position, for $i = 1, \dots, p^r$.

Fix

$$q = (\overbrace{c, \dots, c}^{n_1}, \overbrace{c^2, \dots, c^2}^{n_2}, \overbrace{c^3, \dots, c^3}^{n_3}, \dots, \overbrace{c^{k-1}, \dots, c^{k-1}}^{n_{k-1}}, 1, 1, \dots, 1) \in N,$$

where the first n_1 -entries are c , the $(n_1 + 1)$ -th entry to the $(n_1 + n_2)$ -th is c^2 and so for.

Let Q be the subgroup of N generated by q . Observe that Q is a cyclic subgroup of N of order p^t . Let $\delta \in \text{Lin}(Q)$ such that $\delta(q) = \alpha(c)$. Since $p^t \geq k$ and $\alpha(c)$ is a primitive p^t -root of unity, then for $1 \leq i, j \leq k-1$ we have that $\delta^i \neq \delta^j$ if $i \neq j$. Observe that $\lambda_l(q) = \alpha(c) = \delta(q)$ for $1 \leq l \leq n_1$ and so $n_1 = (\psi_Q, \delta)$. For $l > n_1$, we can check that $\lambda_l(q) = \alpha(c^j) = \alpha^j(c) = \delta^j(q)$ if and only if $\sum_{i=1}^{j-1} n_i < l \leq \sum_{i=1}^j n_i$ and so $n_j = (\psi_Q, \delta^j)$. Observe that if $l > \sum_{i=1}^{k-1} n_i$, then $\lambda_l(q) = \alpha(1) = 1$ and so $n_k = p^r - \sum_{i=1}^{k-1} n_i = (\psi_Q, 1_Q)$. Thus $\psi_Q = \sum_{i=1}^{k-1} n_i \delta^i + n_k 1_Q$.

Observe that Theorem A follows then by the previous example and Theorem B.

REFERENCES

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